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De Sitter gravity and Liouville theory

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Abstract: We show that the spectrum of conical defects in three-dimensional de Sitter space is in one-to-one correspondence with the spectrum of vertex operators in Liouville conformal field theory. The classical conformal dimensions of vertex operators are equal to the masses of the classical point particles in dS_3, that cause the conical defect. The quantum dimensions instead are shown to coincide with the mass of the Kerr-dS_3 solution computed with the Brown-York stress tensor. Therefore classical de Sitter gravity encodes the quantum properties of Liouville theory. The equality of the gravitational and the Liouville stress tensor provides a further check of this correspondence. The Seiberg bound for vertex operators translates on the bulk side into an upper mass bound for classical point particles. Bulk solutions with cosmological event horizons correspond to microscopic Liouville states, whereas those without horizons correspond to macroscopic (normalizable) states. We also comment on recent criticisms by Dyson, Lindesay and Susskind, and point out that the contradictions found by these authors may be resolved if the dual CFT is not able to capture the thermal nature of de Sitter space. Indeed we find that on the CFT side, de Sitter entropy is merely Liouville momentum, and thus has no statistical interpretation in this approach.

Keywords: AdS-CFT and dS-CFT Correspondence, Models of Quantum Gravity.

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Contents

1. Introduction

Recently there has been an increasing interest in gravity on de Sitter (dS) spacetimes [1, 2]. The motivation for this comes partially from recent astrophysical data that point towards a positive cosmological constant, but apart from phenomenological aspects, it also remains an outstanding challenge to understand the role of de Sitter space in string theory, to clarify the microscopic origin of de Sitter entropy [3],[1] and to study in which way the holographic principle [11] is realized in the case of de Sitter gravity. Whereas string theory on anti-de Sitter spaces is known to have a dual description in terms of certain superconformal field theories [12], such an explicit duality emerging from string theory is still missing for dS spacetimes. Yet, based on [13], where the first evidence for a dS/CFT correspondence was given, and on related ideas that appeared in [14, 1], Strominger proposed recently a holographic duality relating quantum gravity on dS_D to a euclidean conformal field theory residing on the past (or alternatively future) boundary $\mathcal{I}^-$ ($\mathcal{I}^+$) of dS_D [15]. Subsequently, this correspondence was further explored in [16]–[23].

In general, the conformal field theories in question seem to be non-unitary. Indeed, if the bulk fields, to which CFT operators couple, become sufficiently massive, the conformal weights of these operators turn out to be complex [15]. In a certain sense, this is puzzling, because the bulk fields provide unitary representations of the de Sitter isometry group $SO(D, 1)$ for arbitrarily large masses [24].

A full understanding of the proposed dS/CFT correspondence seems to require an embedding of de Sitter space in string theory. De Sitter solutions in ordinary supergravity theories [25] break of course all supersymmetries, and this makes it questionable how far

[1]For microscopic derivations of dS entropy based on the Chern-Simons formulation of $2 + 1$ dimensional dS gravity, or on other approaches that are not directly related to string theory, cf. [4]–[10].

one could trust a Maldacena-type argument in this case. On the other hand, Hull's IIB* theory [13] admits a supersymmetric $dS_5 \times H^5$ vacuum, but unfortunately the theory has ghosts.

While at present we do not yet have a satisfactory comprehension of these issues, we can still try to get some insight by studying the simplest explicit example of a dS/CFT correspondence, namely the one between pure dS gravity in three dimensions, and euclidean Liouville theory [17]. This is the aim of the present paper.

We will see that Liouville field theory is able to capture many features of 3d gravity with positive cosmological constant. First of all, it reproduces correctly Strominger's central charge $c = 3l/2G$ [15][2] that appears in the asymptotic symmetry algebra of dS_3 gravity. Furthermore, we show that the spectrum of Liouville vertex operators is in one-to-one correspondence with the spectrum of bulk (gravity) solutions. While the classical conformal dimensions of these operators reproduce exactly the mass of the point particle sources that are present on the 3d gravity side, their quantum dimensions coincide with the mass of the bulk solutions computed with the Brown-York stress tensor. Classical de Sitter gravity thus encodes the quantum properties of Liouville theory. We shall see that features of Liouville theory, like the Seiberg bound [26], as well as the appearance of macroscopic (normalizable) and microscopic (non-normalizable) states [26] do all have an interpretation in 3d de Sitter gravity. In particular, bulk solutions with cosmological event horizons correspond to microscopic Liouville states, whereas those without horizons correspond to macroscopic states. Besides, complex conformal weights occur naturally in Liouville theory, which is exactly what seems to be required in a CFT dual to dS gravity.

The remainder of our paper is organized as follows: in section 2 we briefly review the Kerr-de Sitter solution in three dimensions, and compute the associated Brown-York boundary stress tensor and the conserved charges like mass and angular momentum. Furthermore, we discuss the thermodynamics of the solution, and argue that a consistent thermodynamics can be formulated only in absence of conical defects. We observe that the basic obstacle in understanding thermodynamics is rooted in the absence of a zero temperature state. In section 3 we introduce other configurations of de Sitter gravity, just to remind ourselves and the reader that Liouville field theory has to face many other problems before one can say it is a successful description of quantum three-dimensional gravity. In section 4, starting from the Liouville action obtained in [17], we compute the central charge and show that it coincides with the one obtained in [15]. The Liouville field Φ corresponding to the Kerr-dS_3 solution is then determined. This leads to a relation between Liouville vertex operators $e^{\alpha\Phi}$ and bulk solutions. We show that the Liouville stress tensor equals the Brown-York stress tensor of dS_3 gravity, and that the quantum dimensions of vertex operators reproduce exactly the masses of the bulk solutions. After that, it is shown what normalizable states and the Seiberg bound in Liouville theory correspond to on the bulk side. In section 5, we comment on recent criticisms by Dyson, Lindesay and Susskind [27], and point out that the contradictions found by these authors may be resolved if the dual CFT is not able to capture the thermal nature of de Sitter space. Indeed we find that on

[2]Cf. also [16] for alternative derivations of the central charge.

the CFT side, de Sitter entropy is merely Liouville momentum, and thus has no statistical interpretation in this approach. We conclude with some final remarks.

2. The Kerr-de Sitter solution

Consider the Einstein-Hilbert action with positive cosmological constant $\Lambda = l^{-2}$,

$$I = \frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\sqrt{-g}\left(R - \frac{2}{l^2}\right) + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^2x\sqrt{\gamma}K\,, \tag{2.1}$$

where we included the Gibbons-Hawking boundary term necessary to have a well-defined variational principle. K is the trace of the extrinsic curvature $K_{\mu\nu} = -\nabla_{(\mu}n_{\nu)}$ of the spacetime boundary $\partial\mathcal{M}$, with n^μ denoting the outward pointing unit normal. γ is the induced metric on the boundary.

The equations of motion following from (2.1) admit the Kerr-de Sitter solution given by

$$ds^2 = -N^2dt^2 + N^{-2}dr^2 + r^2(d\phi + N^\phi dt)^2\,, \tag{2.2}$$

with the lapse and shift functions

$$N^2 = \mu - \frac{r^2}{l^2} + \frac{16G^2J^2}{r^2}\,, \qquad N^\phi = \frac{4GJ}{r^2}\,. \tag{2.3}$$

Eq. (2.2) describes the gravitational field of a point particle with spin J, and mass related to the parameter μ (cf. below). The Kerr-dS spacetime has a cosmological event horizon located at[3]

$$r = r_+ = \frac{l}{2}(\sqrt{\tau} + \sqrt{\bar{\tau}})\,, \tag{2.4}$$

where we defined

$$\tau = \mu - \frac{8GJi}{l}\,. \tag{2.5}$$

The Bekenstein-Hawking entropy, temperature, and angular velocity of the horizon are given respectively by

$$\begin{aligned} S &= \frac{2\pi r_+}{4G} = \frac{\pi l}{4G}(\sqrt{\tau} + \sqrt{\bar{\tau}})\,, \\ 2\pi l T &= \frac{r_+}{l} + \frac{16G^2J^2 l}{r_+^3}\,, \\ \Omega &= -\frac{4GJ}{r_+^2}\,. \end{aligned} \tag{2.6}$$

In order to compute the mass and angular momentum of the spacetime (2.2) one can proceed as in [3], and integrate the Killing identity in the static patch from $r = 0$ to $r = r_+$. In doing this, one has to take into account the delta function sources at $r = 0$. An alternative approach, which we will pursue here, makes use of the conserved stress tensor [28] associated to the boundary $\partial\mathcal{M}$. In our setting, $\partial\mathcal{M}$ is the past boundary $\mathcal{I}^-$

[3]The horizon exists for every $\mu \in \mathbb{R}$ if $J \neq 0$, and for positive μ if $J = 0$.

($r \to \infty$) of the conformal compactification of the Kerr-dS$_3$ spacetime, and the stress tensor derived from the action (2.1) reads [15]

$$T^{\mu\nu} = \frac{1}{8\pi G}\left[K^{\mu\nu} - K\gamma^{\mu\nu} - \frac{1}{l}\gamma^{\mu\nu}\right], \tag{2.7}$$

where the last term comes from a surface counterterm added to the action (2.1) in order to cancel divergences [15].

For the metric (2.2), a straightforward calculation yields for $r \to \infty$

$$T_{tt} = -\frac{\mu}{16\pi G l}, \qquad T_{t\phi} = \frac{J}{2\pi l}, \qquad T_{\phi\phi} = \frac{\mu l}{16\pi G}. \tag{2.8}$$

Note that (2.8) is traceless, $h^{\mu\nu}T_{\mu\nu} = 0$, where $h_{\mu\nu}dx^\mu dx^\nu = dt^2 + l^2 d\phi^2$ denotes the CFT metric. In complex coordinates $z = \phi + it/l$, the energy-momentum tensor reads

$$T_{zz} \equiv T(z) = \frac{l\tau}{32\pi G}, \qquad T_{\bar{z}\bar{z}} \equiv \overline{T}(\bar{z}) = \frac{l\bar{\tau}}{32\pi G}, \qquad T_{z\bar{z}} = 0. \tag{2.9}$$

For later use, we also need the stress tensor on the plane. One can pass from the cylinder to the plane (with coordinates $w, \bar{w}$) by setting

$$w = e^{iz}. \tag{2.10}$$

This leads to

$$T(w) = \frac{1-\tau}{4\pi\gamma^2 w^2}, \qquad \overline{T}(\bar{w}) = \frac{1-\bar{\tau}}{4\pi\gamma^2 \bar{w}^2}, \qquad T_{w\bar{w}} = 0, \tag{2.11}$$

where the τ- and $\bar{\tau}$-independent parts come from the schwartzian derivatives.

To each bulk Killing vector field ξ one can associate a conserved charge Q_ξ [28],

$$Q_\xi = \int_0^{2\pi} T_{\mu\nu} u^\mu \xi^\nu \sqrt{\sigma} d\phi, \tag{2.12}$$

where $u = (-N^2)^{-1/2}(\partial_t - N^\phi \partial_\phi)$ is the unit normal to the surface Σ_t of constant t in $\partial\mathcal{M}$, and σ denotes the induced metric on Σ_t. One then gets for the mass Q_{∂_t} and angular momentum Q_{∂_ϕ}

$$Q_{\partial_t} = -\frac{\mu}{8G} =: M, \qquad Q_{\partial_\phi} = J, \tag{2.13}$$

confirming that the parameter J represents the angular momentum of the system.

Consider now the case of vanishing angular momentum, $J = 0$. By the coordinate transformation

$$\begin{aligned} r &= \sqrt{\mu}\, l \cosh\tau \sin\theta, \\ t &= \frac{l}{\sqrt{\mu}} \operatorname{artanh}\left(\frac{\tanh\tau}{\cos\theta}\right), \\ \phi &= \frac{\varphi}{\sqrt{\mu}}, \end{aligned} \tag{2.14}$$

the metric can be cast into the form

$$ds^2 = -l^2 d\tau^2 + l^2 \cosh^2\tau (d\theta^2 + \sin^2\theta d\varphi^2). \tag{2.15}$$

Transformation (2.14) is valid inside the static patch $0 \leq r < r_+$. For $r > r_+$ one has to use instead

$$t = \frac{l}{\sqrt{\mu}} \operatorname{artanh}(\coth\tau\cos\theta)\,, \tag{2.16}$$

and the transformation formulas for r and ϕ remain unchanged.

Eq. (2.15) describes dS_3 as a contracting and expanding two-sphere, with global coordinates τ, θ, φ. We have to take into account however that φ is identified modulo $2\pi\sqrt{\mu}$, so one has a deficit angle $2\pi(1-\sqrt{\mu})$ for $\mu < 1$, and an excess angle $2\pi(\sqrt{\mu}-1)$ for $\mu > 1$. One might assume that excess angles lead to inconsistencies. Yet, as we shall see in section 4, excess angles imply negative conformal weights of the corresponding vertex operators in the dual Liouville field theory. As is well-known, this is not a problem in Liouville theory, where the spectrum of conformal dimensions is not bounded from below [26].

Now we would like to discuss an important feature of the solutions so far discussed. The euclidean section of the spinless defects suggests that euclidean time is a periodic variable with period $\beta = 2\pi l/\sqrt{\mu}$. The Wightman functions will share this periodicity, so an Unruh detector will register a local temperature $T_{\mathrm{loc}} = (-g_{tt})^{-1/2}\beta^{-1}$. However, this does not necessarily mean that there exists a thermal equilibrium state regular everywhere, because the metric is still singular at the conical defect located in $r = 0$ (this will produce divergences in the stress tensor for matter at any temperature, near $r = 0$) and there is no possibility to obtain a euclidean metric that is everywhere nonsingular.[4]

Anyway, thinking for a while in terms of temperature, one may note that there is no zero temperature state in this case, since the metric (2.2) (with $J = 0$) describes actually a cylindrical universe expanding (or contracting) from (to) a wirelike singularity. The solution is either timelike and null geodesically incomplete in the past or in the future. The future incomplete solution can be written in the form

$$ds^2 = -dT^2 + e^{-2T/l}(dR^2 + l^2 d\phi^2)\,,$$

and the other has the scale factor with positive exponent. The non-existence of a zero temperature state makes it hard to make sense of a statistical partition function, like the familiar $\operatorname{tr}\exp(-\beta H)$.

We will now gain some evidence in favour of a non-thermal interpretation of the partition function for de Sitter gravity, by computing the euclidean action

$$I = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\sqrt{-g}\,(R - 2l^{-2})$$

for the solution (2.2) with $J = 0$. The action is not really well-defined, so we cut a small disk around $r = 0$ with radius ϵ and compute I as a sum of two contributions, one from the disk and the other from the complementary region. At the end we let $\epsilon \to 0$. The bulk contribution trivially has a limit as $\epsilon \to 0$, which is

$$I_0 = -\frac{\pi l\sqrt{\mu}}{2G}\,. \tag{2.17}$$

[4]This is somehow reminiscent of the higher-dimensional case, where one has two horizons, a cosmological one and a black hole horizon.

The disk contribution can be calculated using the Gauss-Bonnet theorem: the lapse function near $r=0$ is $\sqrt{\mu}$, the three-dimensional scalar curvature reduces to the two-dimensional curvature on the disk, so Gauss-Bonnet gives

$$I_{\rm disk} = -\frac{\pi l}{2G}(1-\sqrt{\mu})\,.$$

Since $I = I_0 + I_{\rm disk}$, the μ-dependent terms cancel and we are left with the euclidean action of pure de Sitter space! This indicates that the static, structureless pointlike defect carries no entropy, which entirely comes from the cosmological horizon. Indeed, the disk contribution to the action can be written as $-\beta_{\rm dS} m_{\rm class}$, where $\beta_{\rm dS} = 2\pi l$ is the inverse temperature of de Sitter space and $m_{\rm class} = (1-\sqrt{\mu})/4G$ is the classical mass of the point defect (see below). This also suggests to take $I = I_0$, i.e. one deletes the singular point $r = 0$ from the manifold to obtain a regular one. Interpreted as a thermal partition function, this gives an entropy which is twice as large as the Bekenstein-Hawking entropy. Interpreted as a microcanonical partition function instead, it gives an entropy

$$S = \frac{\pi l\sqrt{\mu}}{2G} = \frac{A}{4G}\,.$$

To check the internal consistency of the microcanonical interpretation, we note that if we knew the mass the temperature could be derived from the Gibbons-Hawking differential mass formula [3] in de Sitter space, which states that

$$\frac{dS}{d(-M)} = \frac{2\pi}{\kappa} = \beta\,,$$

where κ is the surface gravity of the cosmological horizon and M is the mass within the horizon (so $-M$ is the mass outside the horizon). Considering the mass (2.13) we find indeed the correct relation

$$\frac{dS}{d(-M)} = \frac{2\pi l}{\sqrt{\mu}}\,.$$

The canonical partition function I_c is a different object, and is defined as the Legendre transform of I_0,

$$I_c = I_0 - M\frac{dI_0}{dM} = I_0 - \beta M = -\frac{\pi^2 l^2}{2G}\,\beta^{-1}\,.$$

The internal energy in the canonical ensemble is then $U = \partial_\beta I_c = -M$ and the entropy is $A/4G$ again.[5] It is worth noting that by adding to I_c the term βE_c, where $lE_c = -(c+\tilde{c})/24$ is the Casimir energy, one obtains a modular invariant partition function, invariant under $\beta \to 4\pi^2 l^2/\beta$.

Comparing with AdS_3, the situation is clear: there the action is infinite, but we have a zero temperature state to compare with, the difference between the two actions is finite and describes correctly the semi-classical thermodynamics of AdS_3 black holes. In the de Sitter case, the action is already finite but there is no zero temperature state to compare with, so the action has no simple thermodynamical interpretation. Instead, it can be interpreted

[5] For similar conclusions in the case of the Schwarzschild-de Sitter solution in four dimensions cf. [29].

consistently as a microcanonical partition function. In this case, $\exp(-I)$ is directly related to the probability of a de Sitter configuration with given energy (see, e.g. [30] and references therein).

Below we will propose an alternative description of de Sitter entropy in Liouville field theory. We shall see that S is essentially the Liouville momentum, and has thus no statistical interpretation in this approach.

3. Other configurations

The Kerr-de Sitter solution is of course not unique, three-dimensional gravity with positive cosmological constant $\Lambda = l^{-2}$ having a lot of other solutions. The only degree of freedom of the spherical sector of 3d gravity is the total volume $\int \sqrt{g^{(2)}}$ of the space. For different topological sectors there is more than this, the moduli of the spatial metric being a complete set of configuration variables [31]–[34]. We present here the general vacuum solution with toroidal topology, and confirm that this also leads to a central charge $3l/2G$. This will amount to show that the asymptotic behaviour at past or future infinity is locally the same as for de Sitter gravity.

In a gauge where the spatial metric depends only on time,

$$ds^2 = -dt^2 + g_{ij}(t)dx^i dx^j\,, \qquad x^i = (x, y)\,,$$

the field equations (in a self-explanatory matrix notation) are

$$\partial_t \operatorname{tr}(g^{-1}\dot{g}) + \frac{1}{2}\operatorname{tr}(g^{-1}\dot{g}g^{-1}\dot{g}) = 4\Lambda\,, \tag{3.1}$$

$$\partial_t (g^{-1}\dot{g}) + \frac{1}{2}g^{-1}\dot{g}\operatorname{tr}(g^{-1}\dot{g}) = 4\Lambda\,. \tag{3.2}$$

One consequence is the identity expressing the hamiltonian constraint of general relativity,

$$\operatorname{tr}(g^{-1}\dot{g})\operatorname{tr}(g^{-1}\dot{g}) - \operatorname{tr}(g^{-1}\dot{g}g^{-1}\dot{g}) = 8\Lambda\,. \tag{3.3}$$

It is readily verified that (3.3) is equivalent to the statement that $\det(g^{-1}\dot{g}) = 4\Lambda$. In this case

$$S = \frac{l}{2}g^{-1}\,\dot{g} \in \mathrm{SL}(2,\mathbb{R})\,.$$

The Einstein equations admit then an $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ symmetry, $g \to UgV^{-1}$, under which the matrix S transform as $S \to VSV^{-1}$. Since we are assuming that space is a torus, the general solution must depend on four integration constants, corresponding to the two moduli of a torus and their conjugate momenta. We can encode these constants into the periods of the spatial variables, which we assume to be identified according to

$$(x, y) \simeq (x + q_1, y + q_2) \simeq (x + q_3, y + q_4)\,.$$

It is not hard to verify that the general solution of the field equations is (almost, but not quite)

$$ds^2 = -dt^2 + \cosh^2\left(\frac{t}{l}\right)dx^2 + \sinh^2\left(\frac{t}{l}\right)dy^2\,, \tag{3.4}$$

with the given identifications for the pair (x, y). There is also a special solution

$$ds^2 = -dt^2 + e^{\pm 2t/l}|dx + \tau dy|^2\,, \qquad \tau = \tau_1 + i\tau_2\,, \tag{3.5}$$

this time with (x, y) ranging over the unit square. τ with $\tau_2 > 0$ is the modulus of the torus. Passing to the new variables (θ_1, θ_2) given by $x = q_1\theta_1 + q_2\theta_2$, $y = q_3\theta_1 + q_4\theta_2$, the metric (3.4) can also be cast in the form

$$ds^2 = -dt^2 + \left(q_1^2 \cosh^2\left(\frac{t}{l}\right) + q_3^2 \sinh^2\left(\frac{t}{l}\right)\right)|d\theta_1 + \tau(t)d\theta_2|^2\,,$$

where $\tau(t)$ is a time dependent modulus which goes over to a constant τ in the limits $t \to \pm\infty$. The special solution (3.5) is thus the asymptotic limit of the general solution. In both cases the metric approaches

$$ds^2 = -dt^2 + \frac{1}{4}e^{-2t/l}dzd\bar{z}$$

asymptotically at $t \to -\infty$. Note also that the solutions either expand from a wirelike singularity to an infinite volume torus, or either collapse from an infinite torus to a wirelike singularity. So there is only one conformal boundary in this case. This observation is pertinent to Strominger's proposal that there is actually only one CFT for three-dimensional de Sitter gravity [15].

The given asymptotic form is locally the same as the de Sitter case (globally not, of course, the conformal Killing group of the torus is two-dimensional), so Strominger's analysis can be applied: the central charge must be given by $3l/2G$. The boundary stress tensor (2.7) has vanishing trace, which is consistent with the fact that the scalar curvature of the torus vanishes.

Now we observe that there cannot be globally static solutions, because the hamiltonian constraint would give the curvature of the torus as twice the cosmological constant. This contradicts the fact that the Euler characteristic vanishes. There should also be no locally static solutions (we have not proved this), as the space volume always end or begins in a wirelike singularity.[6] The torus Liouville field theory must then be compared with the dynamics, a challenge for the dS_3/CFT_2 correspondence. The difficulty is that the Liouville equation appears as the hamiltonian constraint for Einstein gravity for a spatial metric which is conformal to a constant curvature metric. Thus, classically, the correspondence would mean that the dynamics reduces to the solution of the constraint equations. It was pointed out by Witten [32] that there is a gauge, in the Chern-Simon formulation of 3d gravity, in which the field equations reduce indeed to the constraints, but such a gauge cannot be applied to Einstein equations.

4. dS_3 gravity and Liouville theory

In [17] it was shown that the asymptotic dynamics of pure de Sitter gravity in three dimensions is described by euclidean Liouville field theory. This correspondence was established

[6]There is, however, a cosmological horizon, so the possibility exists for a coordinate transformation to a locally static region.

by writing dS_3 gravity as an $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons theory, with action

$$I = \frac{is}{4\pi}\int \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) - \frac{is}{4\pi}\int \mathrm{Tr}\left(\bar{A}\wedge d\bar{A} + \frac{2}{3}\bar{A}\wedge\bar{A}\wedge\bar{A}\right), \tag{4.1}$$

where, in the conventions of [17],

$$s = -\frac{l}{4G}, \tag{4.2}$$

and

$$A = A^a\tau_a = \left(\omega^a + \frac{i}{l}e^a\right)\tau_a, \qquad \bar{A} = \bar{A}^a\tau_a = \left(\omega^a - \frac{i}{l}e^a\right)\tau_a, \quad a = 0,1,2, \tag{4.3}$$

where e^a is the dreibein, $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{bc}$ the spin connection, and the τ_a are the $\mathrm{SL}(2,\mathbb{C})$ generators given in [17].

Chern-Simons theory is known to reduce to a WZNW model in presence of a boundary [35]. The boundary conditions for asymptotically past de Sitter spaces [15] provide then the constraints for a hamiltonian reduction from the WZNW model to Liouville field theory.

The Liouville action obtained in [17] reads

$$I = -\frac{1}{8\pi G}\int d\phi dt\left[\frac{1}{2}\partial_z\Phi\partial_{\bar{z}}\Phi + 2\exp\Phi\right], \tag{4.4}$$

where we used the complex coordinate $z = \phi + it/l$. In order to conform with the conventions of [26], we rescale $\Phi \to \gamma\Phi$, with $\gamma = \sqrt{8G/l}$. This leads to the action

$$I = -\frac{1}{4\pi}\int \sqrt{h}\, d^2x\left[\frac{1}{2}h^{ij}\partial_i\Phi\partial_j\Phi + \frac{\lambda}{2\gamma^2}\exp(\gamma\Phi)\right], \tag{4.5}$$

where the "cosmological constant" λ is given by $\lambda = 16l^{-2}$, and the CFT metric on the cylinder is $h_{ij}dx^i dx^j = dt^2 + l^2 d\phi^2$. (4.5) coincides, up to the overall sign, with the expression given in [26]. There are some subtleties concerning this sign difference. First of all, when deriving (4.5), one starts from a lorentzian gravitational (or Chern-Simons) action, end ends up with a euclidean action. This usually involves an additional minus sign, which would eliminate the minus in front of (4.5). Presumably these sign ambiguities are also related to the different definitions of the stress tensor used in [15, 16] and [18], that differ by an overall sign. The former implies a positive central charge, whereas the latter leads to $c = -3l/2G$ [18]. Some discussion of these ambiguities can be found in [18]. It would certainly be desirable to understand these better. For the time being, we use the plus sign in front of (4.5). It will turn out that this leads in fact to results that make sense physically. First of all, we can check that the value of the central charge in Liouville theory matches perfectly with that computed from the gravitational side. We have only to remember that the action (4.5) was obtained in a gauge with flat fiducial metric h_{ij}: changing to a conformal gauge, $h_{ij} \to g_{ij} = e^{2u}h_{ij}$, the Liouville field changes from $\gamma\Phi$ to $\gamma\Phi - 2u$. Substituting into eq. (4.5) (with a plus sign in front) and observing that $R = -e^{-2u}\nabla^2 u$, one obtains the action (modulo a field independent term)

$$I = \frac{1}{4\pi}\int \sqrt{g}\, d^2x\left[\frac{1}{2}g^{ij}\partial_i\Phi\partial_j\Phi + \frac{\lambda}{2\gamma^2}\exp(\gamma\Phi) + \frac{Q}{2}R\Phi\right], \tag{4.6}$$

where

$$Q = \sqrt{\frac{l}{2G}}$$

is the background charge. As is well-known, this theory has a classical central charge $c = 3Q^2$, which is equal to the de Sitter charge $3l/2G$. This is positive, as it should be, when one uses the conventions of [15, 16].

As a further consistency check, we will show below that the Liouville stress tensor reproduces the gravitational energy-momentum tensor (2.7), which was defined with the conventions of [15, 16]. Note also that in this way we obtain for the classical Liouville central charge

$$c = \frac{12}{\gamma^2}\,, \tag{4.7}$$

confirming the classical relation of Liouville theory $Q = 2/\gamma$. For the Kerr-dS$_3$ solution (2.2), the Chern-Simons connection (4.3) has the asymptotic behaviour for $r \to \infty$

$$\begin{aligned} A_r &= \begin{pmatrix} \frac{1}{2r} & 0 \\ 0 & -\frac{1}{2r} \end{pmatrix}, \qquad & A_z &= \begin{pmatrix} 0 & \frac{il\tau}{4r} \\ \frac{ir}{l} & 0 \end{pmatrix}, \qquad & A_{\bar z} &= 0\,, \\ \bar A_r &= \begin{pmatrix} -\frac{1}{2r} & 0 \\ 0 & \frac{1}{2r} \end{pmatrix}, \qquad & \bar A_{\bar z} &= \begin{pmatrix} 0 & -\frac{ir}{l} \\ -\frac{il\bar\tau}{4r} & 0 \end{pmatrix}, \qquad & \bar A_z &= 0\,. \end{aligned} \tag{4.8}$$

Following [17], it is straightforward to compute the corresponding Liouville solution

$$e^{\gamma\Phi} = \frac{\tau\bar\tau}{4}\left[\bar\omega e^{\frac{i}{2}(\sqrt{\tau}z+\sqrt{\bar\tau}\bar z)} + \omega e^{-\frac{i}{2}(\sqrt{\tau}z+\sqrt{\bar\tau}\bar z)} - u e^{\frac{i}{2}(\sqrt{\tau}z-\sqrt{\bar\tau}\bar z)} - v e^{-\frac{i}{2}(\sqrt{\tau}z-\sqrt{\bar\tau}\bar z)}\right]^{-2}, \tag{4.9}$$

where $\omega \in \mathbb{C}$ and $u, v \in \mathbb{R}$ denote arbitrary constants satisfying $\omega\bar\omega - uv = |\tau|/4$.[7] If we set $\omega = \rho\sqrt{|\tau|}\exp(i\zeta)$, $\rho \geq 0$, and translate

$$\sqrt{\tau}z \to \sqrt{\tau}z + \zeta + i\ln\frac{u}{\sqrt{|\tau|}(r - 1/2)}\,, \tag{4.10}$$

we obtain

$$\begin{aligned} e^{\gamma\Phi} = \frac{|\tau|}{4}\Bigg[&\rho e^{\frac{i}{2}(\sqrt{\tau}z+\sqrt{\bar\tau}\bar z)} + \rho e^{-\frac{i}{2}(\sqrt{\tau}z+\sqrt{\bar\tau}\bar z)} - \\ &- \left(\rho - \frac{1}{2}\right) e^{\frac{i}{2}(\sqrt{\tau}z-\sqrt{\bar\tau}\bar z)} - \left(\rho + \frac{1}{2}\right) e^{-\frac{i}{2}(\sqrt{\tau}z-\sqrt{\bar\tau}\bar z)}\Bigg]^{-2}. \end{aligned} \tag{4.11}$$

Let us now specify to the case $J = 0$ and $\mu \geq 0$. We choose $\rho = 0$ in (4.11). This yields the ϕ-independent solution

$$e^{\gamma\Phi} = \mu\left[e^{\frac{i}{2}\sqrt{\mu}(z-\bar z)} - e^{-\frac{i}{2}\sqrt{\mu}(z-\bar z)}\right]^{-2}. \tag{4.12}$$

Now remember that the asymptotic boundary of the spacetime (2.2) has the topology of a cylinder. One can pass to the plane (with coordinates $w, \bar w$) by the transformation (2.10), leading to

$$e^{\gamma\Phi}\,dzd\bar z = \frac{\mu}{(w\bar w)^{1-\sqrt{\mu}}\left[1 - (w\bar w)^{\sqrt{\mu}}\right]^2}\,dwd\bar w\,, \tag{4.13}$$

which is the standard classical elliptic solution of Liouville theory [26].

[7] This equality comes from the requirement that the group element $g \in \mathrm{SL}(2,\mathbb{C})$ that appears in the WZNW action, and whose Gauss decomposition contains Φ [17], must have unit determinant.

Semiclassically, the Liouville vertex operators $e^{\alpha\Phi}$ appear as sources of curvature in the classical equation of motion and lead to solutions with local elliptic monodromy with $\sqrt{\mu} = 1 - \gamma\alpha$ [26]. From this we obtain

$$\alpha = \frac{1 - \sqrt{\mu}}{\gamma}\,, \tag{4.14}$$

i.e., a relation between the mass parameter μ of the dS_3 solution and the parameter α of the vertex operator.[8] In the classical theory, $e^{\alpha\Phi}$ has conformal dimension

$$\Delta_{\text{class}}(e^{\alpha\Phi}) = \frac{\alpha}{\gamma} = \frac{1 - \sqrt{\mu}}{\gamma^2} = \frac{l}{8G}(1 - \sqrt{\mu})\,. \tag{4.15}$$

Now the Schwarzschild-de Sitter solution contains a pair of conical defects at antipodal points on the spatial two-sphere, so only one of these is inside the cosmological horizon. From the Gauss-Bonnet theorem applied to the upper hemisphere containing the point $r = 0$, we obtain

$$\int R^{(2)} = 4\pi(1 - \sqrt{\mu}) + C\,,$$

where $R^{(2)}$ denotes the spatial curvature scalar and C is a constant (the integral of $R^{(2)}$ over the space as if there were no defect). This means that there is a curvature singularity with strength

$$R^{(2)}_{\text{sing}} = 4\pi(g^{(2)})^{-1/2}(1 - \sqrt{\mu})\delta(\vec{r})\,.$$

The hamiltonian constraint for a static solution is[9]

$$R^{(2)} - 2\Lambda = 16\pi G T^{00}\,, \tag{4.16}$$

so by comparison we get

$$\sqrt{g^{(2)}}\, T^{00} = \frac{1}{4G}(1 - \sqrt{\mu})\delta(\vec{r})\,.$$

In general, the stress-energy tensor for pointlike masses located at isolated points $\vec{r}_i$ is $\sqrt{g^{(2)}}\, T^{00} = \sum_i m_i \delta(\vec{r} - \vec{r}_i)$, so we learn that the bare mass of the conical defect is

$$m_{\text{class}} = \frac{1 - \sqrt{\mu}}{4G}\,. \tag{4.17}$$

Comparing this with (4.15), one obtains

$$l m_{\text{class}} = \Delta_{\text{class}} + \bar{\Delta}_{\text{class}}\,, \tag{4.18}$$

so the classical conformal weights reproduce exactly the mass of the classical point particle in dS_3 that causes the conical defect.

The quantum dimension of the operator $e^{\alpha\Phi}$ is given by

$$\Delta(e^{\alpha\Phi}) = \frac{\alpha}{\gamma} - \frac{1}{2}\alpha^2 + \mathcal{O}(1) = \frac{1 - \mu}{2\gamma^2} = \frac{c}{24}(1 - \mu) = \frac{l}{16G}(1 - \mu)\,. \tag{4.19}$$

[8]In the AdS_3 case, an interpretation of Liouville non-normalizable states in terms of particles moving in the bulk was developed in [36].

[9]The stress tensor appearing in eq. (4.16) is not to be confused with (2.7).

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Using $\bar{\Delta} = \Delta$ (the vertex operators are scalars), and $\tilde{c} = c$, we obtain

$$\Delta + \bar{\Delta} = lM + \frac{c + \tilde{c}}{24}\,, \tag{4.20}$$

so that $(\Delta + \bar{\Delta})/l$ coincides (modulo a constant shift of $(c + \tilde{c})/24l$ coming from the transformation (2.10) from the cylinder to the plane) with the mass M that we computed in (2.13).

This mass includes the contribution of the gravitational field, and should not be confused with the bare mass of the particles that cause the conical defects and which act as static external conditions. In particular, since μ must be strictly positive to have a static solution, the bare masses should be bounded from above, a fact that will be confirmed below from the Seiberg bound. If we like, we can think of the mass that it is a kind of gravitational dressing effect.

We have thus established a one to one correspondence between Liouville vertex operators in the boundary CFT and de Sitter solutions in the bulk. The quantum conformal weights of the operators are identical to the masses of the bulk solutions. Also, as already noticed above, the spectrum of conformal dimensions in Liouville theory, unlike the one of a generic CFT, is not bounded from below. Instead one has $\Delta \leq Q^2/8$ [26], where Q denotes the background charge,[10] $Q = 2/\gamma$. This leads to the inequality $\mu \geq 0$, but otherwise μ is unrestricted. In particular, values $\mu > 1$ corresponding to excess angles, yield $\Delta < 0$, and thus seem to be entirely consistent from the Liouville point of view. Furthermore, we note that the solution with $\mu = 0$, which is essentially de Sitter space in inflationary coordinates, has $\alpha = 1/\gamma$, i.e., it corresponds to the puncture operator.

A further point to check is the equality of the Liouville stress tensor and the Brown-York energy-momentum tensor (2.11) of three-dimensional de Sitter gravity. For $J = 0$, the latter is given by

$$T(w) = \frac{1-\mu}{4\pi\gamma^2 w^2}\,, \qquad \overline{T}(\bar{w}) = \frac{1-\mu}{4\pi\gamma^2 \bar{w}^2}\,, \tag{4.21}$$

which indeed coincides with the Liouville stress tensor for elliptic solutions [26]. Note also that $T(w)$ can be written as

$$T(w) = \frac{1}{2\pi\gamma^2} S[f(w); w]\,, \tag{4.22}$$

where

$$S[f(w); w] = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \tag{4.23}$$

denotes the schwartzian derivative, and the "uniformizing map" $f(w)$ is given by $f(w) = w^{\sqrt{\mu}}$.

Let us now consider the gravity solutions with $\mu < 0$. We will see that they also have a nice interpretation in Liouville theory.

[10] Strictly speaking, in the quantum theory one has $Q = 2/\gamma + \gamma$. As $\gamma^2 = 8l_P/l$ ($l_P = G$ denotes the Planck length in three dimensions), and we only consider the regime $l \gg l_P$ where one can trust classical gravity, we have $\gamma \ll 1$, so that the corrections to the classical background charge $2/\gamma$ can be neglected.

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Defining E by $\sqrt{\mu} = iE\gamma$, one gets for the classical Liouville solution corresponding to (2.2) with $J = 0$ and $\mu < 0$[11]

$$e^{\gamma\Phi} dz d\bar{z} = \frac{E^2\gamma^2}{4w\bar{w}\sin^2\left(\frac{E\gamma}{2}\ln w\bar{w}\right)} dw d\bar{w}\,, \tag{4.24}$$

which can also be obtained from (4.13) by analytical continuation. As is well-known (cf. e.g. [37] for a review), the hyperbolic solution (4.24) corresponds in the semiclassical limit $\gamma \to 0$ to the normalizable quantum states ψ_E with momentum E, i.e., to the so-called macroscopic states. Formally, these states can be associated to vertex operators $e^{\alpha\Phi}$ with

$$\alpha = \frac{Q}{2} + iE\,. \tag{4.25}$$

If we use $\sqrt{\mu} = iE\gamma$ in our relation (4.14) that connects vertex operators and bulk solutions, we get exactly (4.25). Furthermore, since the quantum state ψ_E has energy $E^2/2 + Q^2/8$ [26], the sum of the energies of ψ_E (right-moving) and ψ_{-E} (left-moving) is

$$E^2 + \frac{Q^2}{4} = \frac{1-\mu}{\gamma^2} = lM + \frac{c+\tilde{c}}{24}\,, \tag{4.26}$$

which gives again the mass (2.13).

Summarizing, one has thus the following picture for $J = 0$: gravity solutions that have a temperature (i.e., with $\mu \geq 0$) correspond to vertex operators with $\alpha = (1-\sqrt{\mu})/\gamma$, i.e., to non-normalizable or microscopic states. Solutions with $\mu < 0$ correspond to normalizable or macroscopic states with real momentum E, where E is given by $\sqrt{\mu} = iE\gamma$. Also the Seiberg bound [26], which states that operators with $\alpha > Q/2$ do not exist, has an interpretation in de Sitter gravity. Using (4.15) and $lm_{\text{class}} = \Delta_{\text{class}} + \bar{\Delta}_{\text{class}}$, the Seiberg bound implies an upper mass bound $m_{\text{class}} \leq 1/4G$ for classical point particles in de Sitter space.

Up to now we only considered the case of vanishing angular momentum. It is of course natural to ask how to interpret the rotating Kerr-dS$_3$ solution on the CFT side. The first point we note is that one can formally associate the solution to CFT operators with complex conformal weights

$$\Delta = \frac{1-\tau}{2\gamma^2}\,, \qquad \bar{\Delta} = \frac{1-\bar{\tau}}{2\gamma^2}\,, \tag{4.27}$$

in terms of which

$$lM = \Delta + \bar{\Delta} - \frac{c+\tilde{c}}{24}\,, \qquad iJ = \Delta - \bar{\Delta}\,. \tag{4.28}$$

In Liouville theory, one may consider more general vertex operators $e^{\alpha\Phi}$ with $\alpha \in \mathbb{C}$ [39]. These are in one-to-one correspondence with microscopic states if and only if Re$(\alpha) <$

[11]It is interesting to note that the solutions with temperature ($\mu > 0$) have elliptic monodromy, whereas those without temperature ($\mu < 0$) have hyperbolic monodromy. In AdS$_3$ it happens the other way round: The BTZ black hole has hyperbolic monodromy, whereas a particle in AdS$_3$ has elliptic monodromy [38].

$Q/2$ [39], which is the generalized Seiberg bound. If we replace μ by τ in (4.14) for $J \neq 0$, we get for the case of nonvanishing angular momentum

$$\alpha = \frac{1-\sqrt{\tau}}{\gamma}\,, \tag{4.29}$$

which is complex and has $\mathrm{Re}(\alpha) \neq Q/2$, i.e., does not correspond to normalizable states. Using (2.5), one obtains from (4.29)

$$\alpha = \frac{1}{\gamma} \mp \sqrt{\frac{\mu+|\tau|}{2\gamma^2}} \pm \frac{i\gamma J}{\sqrt{2(\mu+|\tau|)}}\,. \tag{4.30}$$

If we choose the upper sign in (4.30), the Seiberg bound $\mathrm{Re}(\alpha) < 1/\gamma$ is satisfied. One can thus make sense of vertex operators $e^{\alpha\Phi}$, with α given by the upper sign expression in (4.30). In the quantum theory, these have conformal weights [39]

$$\Delta = \frac{\alpha}{\gamma} - \frac{1}{2}\alpha^2 = \frac{1-\mu}{2\gamma^2} + \frac{iJ}{2}\,, \tag{4.31}$$

which is exactly (4.27). In conclusion, we showed that the Kerr-de Sitter solution with nonvanishing angular momentum can be associated to vertex operators $e^{\alpha\Phi}$ with α as in (4.30) with the upper signs.

5. Is the dual CFT thermal?

Recently, some criticisms concerning the existence of a dS/CFT correspondence appeared in the literature [27]. In particular, the authors of [27] considered a general finite closed system described by a thermal density matrix, and a thermal correlator

$$F(t) = \langle \mathcal{O}(0)\mathcal{O}(t)\rangle\,. \tag{5.1}$$

It was then shown that the long time average of $F(t)F^*(t)$ is non-zero and positive, which leads to a contradiction with the dS/CFT result in static coordinates [16],[12]

$$\langle \mathcal{O}(0,0)\mathcal{O}(t,\phi)\rangle \sim \left[\cosh\frac{t}{l} - \cos\phi\right]^{-h}\,, \tag{5.2}$$

where $h = 1 + \sqrt{1-m^2l^2}$. (5.2) is not the standard thermal correlator, rather it is the two-point function for dimension (h,h) operators on a cylinder, whose length (not circumference) is parametrized by the euclidean time coordinate t. (5.2) behaves like

$$\langle \mathcal{O}(0,0)\mathcal{O}(t,\phi)\rangle \sim e^{-ht/l} \tag{5.3}$$

for large t. Clearly the behaviour (5.3) would imply a zero long-time average. Obviously the apparent contradiction found in [27], which is based on the assumption that the dual CFT is described by a thermal density matrix, is resolved if the conformal field theory does

[12]We only consider the simple case of operators $\mathcal{O}$ that couple to bulk scalars of mass m.

not encode the thermal nature of de Sitter space. We would like to point out here some arguments in favour of this.

First of all, the concept of assigning a temperature to de Sitter space is well-defined only in the static patches. However, the past (and future-) boundary, where the CFT resides, lies outside the static region. In particular, the local Tolman temperature of de Sitter space,

$$T(r) = \frac{1}{2\pi l\sqrt{1 - r^2/l^2}} \tag{5.4}$$

formally becomes imaginary for $r > l$. It might thus be that the conformal field theory on $\mathcal{I}^-$ does not capture the thermal nature of de Sitter space. If this is true, and if the dual CFT nevertheless accounts somehow for de Sitter entropy, we do not expect this entropy to be thermal.[13]

In section 2 we argued that the partition function of dS gravity might have a non-thermal interpretation. We will now confirm that in our Liouville approach this is indeed the case. Let us start with the KPZ equation for gravitational dressing of CFT operators with bare conformal weight Δ_0 by vertex operators $e^{\alpha\Phi}$,

$$\alpha - \frac{Q}{2} = -\sqrt{\frac{1}{4}Q^2 - 2 + 2\Delta_0}\,. \tag{5.5}$$

Setting $\Delta = 1 - \Delta_0$ yields

$$\alpha - \frac{Q}{2} = -\sqrt{\frac{1}{4}Q^2 - 2\Delta}\,, \tag{5.6}$$

which is of course the formula for the quantum conformal weights Δ of vertex operators $e^{\alpha\Phi}$. Now observe that the entropy of the Schwarzschild-dS$_3$ solution ((2.2) with $J = 0$) is given by

$$S = \frac{\pi l\sqrt{\mu}}{2G} = \frac{4\pi\sqrt{\mu}}{\gamma^2}\,, \tag{5.7}$$

and, using (4.14), that

$$\alpha - \frac{Q}{2} = -\frac{\sqrt{\mu}}{\gamma} = -\frac{S\gamma}{4\pi}\,. \tag{5.8}$$

Inserting (5.8) into (5.6), one obtains

$$S = 2\pi\sqrt{2Q^2\left(\frac{Q^2}{8} - \Delta\right)}\,. \tag{5.9}$$

If we finally use the central charge $c = 3Q^2$ and $\bar{\Delta} = \Delta$, we get

$$S = 2\pi\sqrt{\frac{c}{6}\left(\frac{c}{24} - \Delta\right)} + 2\pi\sqrt{\frac{c}{6}\left(\frac{c}{24} - \bar{\Delta}\right)}\,. \tag{5.10}$$

Eq. (5.10) looks like the Cardy formula for the asymptotic level density of conformal field theories, but actually it is not, because the signs of the terms Δ and $c/24$ are interchanged.

[13] A non-thermal interpretation of de Sitter entropy in terms of a sort of euclidean entanglement entropy was recently proposed in [22].

Rather, we saw that (5.10) is the KPZ equation. Looking at (5.8), one also sees what de Sitter entropy corresponds to in Liouville theory. Using

$$iE = \alpha - \frac{Q}{2} \tag{5.11}$$

for the Liouville momentum E, one finally obtains

$$S = -\frac{4\pi i}{\gamma}E\,, \tag{5.12}$$

so that de Sitter entropy is essentially Liouville momentum. In particular, it has no statistical meaning in this approach. Note that the Schwarzschild-dS$_3$ solution with cosmological horizon corresponds to imaginary momentum E, so that S is real, as it should be.

6. Final remarks

In summary, we established a detailed and quantitative correspondence between three-dimensional gravity with positive cosmological constant and euclidean Liouville theory. Of course the relationship between gravity in three dimensions and Liouville theory at the classical level is rather well-known; in fact the appearance of the Liouville equation in the context of three-dimensional de Sitter gravity was already pointed out in [40]. What is less obvious, and comes as a surprise, is that classical gravity with positive cosmological constant encodes the quantum properties of Liouville theory. Indeed we found that the quantum dimensions of Liouville vertex operators coincide exactly with the masses of the gravity solutions, computed with the Brown-York stress tensor, which is an entirely classical concept. This fits nicely into the usual notion of duality: on one side, we are at a classical level, whereas in the dual theory we are in the quantum regime.

A further point that seems worth mentioning is that all the bulk solutions with cosmological event horizons (i.e., the ones with $J \neq 0$ or with $J = 0$, $\mu > 0$ in (2.2)) correspond to microscopic states in Liouville theory, whereas the solutions without horizons ($J = 0$, $\mu < 0$) correspond to macroscopic (normalizable) states. This may be a pure coincidence, but it would certainly be interesting to understand this better. We note in this context that solutions with event horizon have a (real) entropy S, and thus, since S is related to the Liouville momentum E by (5.12),[14] must have imaginary E, i.e., must correspond to non-normalizable (microscopic) states.

We also note that turning on angular momentum on the bulk side leads to complex conformal weights, and we saw that such conformal weights indeed naturally appear in Liouville theory. This is what one expects from a CFT that is dual to dS gravity.

It would be interesting to check if the Liouville partition function (probably with vertex operator insertions) reproduces the gravitational (bulk) partition function. This is currently under investigation.

[14](5.12) is valid for $J = 0$, but probably there exists an appropriate generalization to the spinning case.

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